Quiz #3 MATH 10A Section: 101 & 105 09/19

Name: FRANCO

1) Find the limits (1 point each)

a) $\lim_{x \to 0} \frac{\sin x}{\sqrt{x}} = \frac{\sin(0)}{\sqrt{0}} = \frac{0}{0}$

by L'Hôpital

$= \lim_{x \to 0} \frac{\cos x}{\frac{1}{2}x^{-1/2}} = \lim_{x \to 0} 2\sqrt{x}\cos x$

$= 2\sqrt{0}\cos(0) = 0$

b) $\lim_{n \to +\infty} \frac{e^{-n}}{n^{3/2}} = \frac{0}{\infty}$ $\left( \lim_{n \to +\infty} e^{-n} = 0 \quad \lim_{n \to +\infty} n^{3/2} = +\infty \right)$

then

$\lim_{n \to +\infty} e^{-n}\left(\frac{1}{n^{3/2}}\right) = 0 \cdot 0 = 0$

$\left( \lim_{n \to +\infty} \frac{1}{n^{3/2}} = 0 \right)$

2) For $f(x) = e^x$ and $g(x) = x^2 - x - 6$

a) Write the formulas for $g \circ f(x)$ and $f \circ g(x)$ (1 point each)

b) Pick one from $g \circ f$, $f \circ g$ and find its first & second derivative (1 point each)

c) Keep your choice from $g \circ f$, $f \circ g$. Find the intervals where is increasing or decreasing. Find as well if the extrema are local minimum or local maximum, and with that information sketch its graph (2 points)

a) $g \circ f(x) = (e^x)^2 - (e^x) - 6 = e^{2x} - e^x - 6$

$f \circ g(x) = e^{x^2 - x - 6}$

b) Since $f'(x) = e^x$, $g'(x) = 2x - 1$, by chain rule

$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) = (2e^x - 1) \cdot e^x \quad \ldots (1)$

$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = e^{x^2-x-6}(2x-1) \quad \ldots (2)$

From (1), applying product rule:

$(g \circ f)''(x) = (2e^x) \cdot (e^x) + (e^x)(2e^x - 1) = e^x(4e^x - 1)$

From (2), applying product rule:

$(f \circ g)''(x) = \left(e^{x^2-x-6}(2x-1)\right) \cdot (2x-1) + (2) \cdot \left(e^{x^2-x-6}\right)$

$= e^{x^2-x-6}\left((2x-1)^2 + 2\right)$

Extrema of $(g \circ f)$ : $(g \circ f)'(x) = 0 \Rightarrow (2e^x - 1) \cdot e^x = 0$. Since $e^x \neq 0$

$\Rightarrow 2e^x - 1 = 0 \Rightarrow e^x = 1/2 \Rightarrow x = \ln(\frac{1}{2})$

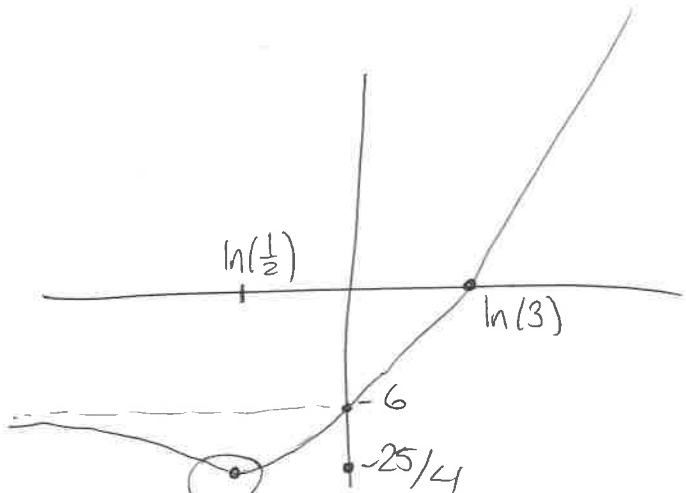
Since $(g \circ f)'(1) = (2e-1) \cdot e > 0$
$(g \circ f)'(-1) = (2e^{-1}-1) \cdot e^{-1} < 0$
$\left( -1 < \ln\frac{1}{2} < 1 \text{ because } 1/e < 1/2 < e \right)$

the sign does not change in those intervals (Bisection)

then $(g \circ f)$ is decreasing in $(-\infty, \ln(\frac{1}{2}))$ and increasing on $(\ln(\frac{1}{2}), +\infty)$

$(g \circ f)(\ln\frac{1}{2}) = (\frac{1}{2})^2 - (\frac{1}{2}) - 6 = -\frac{25}{4}$, $(g \circ f)''(\ln(\frac{1}{2})) = \frac{1}{2}(4 \cdot \frac{1}{2} - 1) = \frac{1}{2} > 0$

then $\ln(\frac{1}{2})$ is a local minimum




local minimum : it is actually global!

$\lim_{x \to +\infty} g \circ f(x) = +\infty$

$\lim_{x \to -\infty} g \circ f(x) = -6$ ← asymptote

$\left( \lim_{x \to -\infty} e^x = 0 \right)$

Extrema of $(f \circ g)$ : $(f \circ g)'(x) = 0 \Rightarrow e^{x^2-x-6}(2x-1) = 0$. Since $e^{x^2-x-6} \neq 0$

$\Rightarrow 2x - 1 = 0 \Rightarrow x = 1/2$

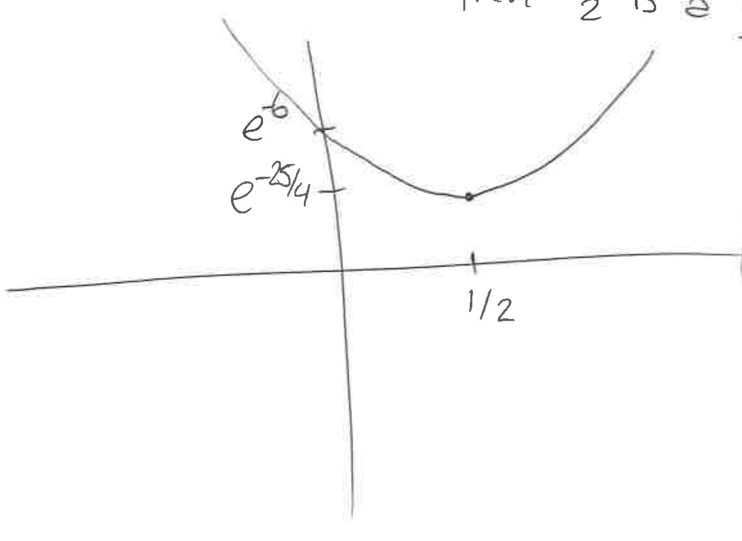
since $(f \circ g)'(1) = e^{-6}(2 \cdot 1 - 1) = e^{-6} > 0$
$(f \circ g)'(0) = e^{-6}(2 \cdot 0 - 1) = -e^{-6} < 0$
$\left( 0 < \frac{1}{2} < 1 \right)$

the sign does not change on those intervals (Bisection)

the $(f \circ g)$ is decreasing in $(-\infty, \frac{1}{2})$ and increasing in $(\frac{1}{2}, +\infty)$

$f \circ g(\frac{1}{2}) = e^{-25/4}$, $(f \circ g)''(\frac{1}{2}) = e^{-25/4}(0^2 + 2) = 2e^{-25/4} > 0$

then $\frac{1}{2}$ is a local minimum




$\lim_{x \to +\infty} f \circ g(x) = +\infty$

$\lim_{x \to -\infty} f \circ g(x) = +\infty$